Assignment 3.

This assignment is due March 14th. If you need more time, ask for an extension (just don't get overwhelmed by homeworks piling up).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Let V be real vector space spanned by the rows of the matrix

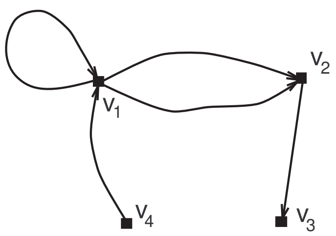
$$\begin{pmatrix} 2 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$$

(a) Find a basis for V.
(b) Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V.
(c) If $(x_1, x_2, x_3, x_4, x_5)$ is in V, what are its coordinates in the basis chosen in part (a)?

(2) Let A be an $m \times n$ matrix over the field F, and consider the system of equations $AX = Y$. Prove that this system of equations has a solution if and only if the row rank of A is equal to the row rank of the augmented matrix of the system (augmented matrix of the system $AX = Y$ is the matrix $A|Y$ that consists of A with an extra column Y).

(3) *(Counting paths in a directed graph via adjacency matrix)* Let Γ be a finite directed graph, that is a finite collection of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges E, each edge $e \in E$ "pointing" from one vertex to another: $v \overset{e}{\to} v'$. (You can think of this as a bunch of points on a plane with arrows drawn from some vertices to others. Multiple arrows between the same vertices and loop edges $v \overset{e}{\to} v$ are allowed.)

Construct the *adjacency matrix* for Γ: it is an $n \times n$ matrix $A = A(\Gamma)$, where each entry A_{ij} is the number of edges that point from v_i to v_j. For example, for the graph



adjacency matrix is

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(a) Consider $A = A(\Gamma)$ and $B = A^2$. Prove that B_{ij} = number of paths (that traverse edges in proper direction) in Γ of length 2 from v_i to v_j. (Length of a path in a graph is the number of edges in this path.) Hint: stare hard at formula $B_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \cdots + A_{in}A_{nj}$.
(b) Prove (for example, using induction) that entry (i,j) in A^n is the number of paths in Γ of length n from v_i to v_j.

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(4) (a) Give an example of 3×3 matrix A such that map $\mathbb{R}^3 \to \mathbb{R}^3$ defined by $X \to AX$ has its range spanned by vectors $(1, 0, 01)$ and $(1, 2, 2)$.
(b) Describe all such matrices. (Hint: what are columns of such a matrix?)

(5) Let V be a vector space and T be a linear transformation from V to V. Prove that the following two statements about T are equivalent.
(a) The intersection of the range of T and the null space of T is trivial.
(b) If $T(T\alpha) = 0$, then $T\alpha = 0$.

(6) Let V be an n-dimensional vector space over the field F, and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be an ordered basis for V.
(a) There is a unique linear operator T on V such that

$$T\alpha_j = \alpha_{j+1}, \quad j = 1, 2, \dots, n-1, \quad T\alpha_n = 0.$$

What is the matrix A of T in the ordered basis α?
(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.
(c) Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis β for V such that matrix of S in β is the matrix A from part (a). (Hint: if $S^n = 0$ but $S^{n-1} \neq 0$, that means that there is a specific vector $v \in V$ such that $S^{n-1}v \neq 0$. Start by taking $\beta_1 = v$.)